Separation of variables in the nonlinear wave equation

1994 J. Phys. A: Math. Gen. 27 L291

(http://iopscience.iop.org/0305-4470/27/9/009)

J. Phys. A: Math. Gen. 27 (1994) L291–L297. Printed in the UK

LETTER TO THE EDITOR

Separation of variables in the nonlinear wave equation

R Z Zhdanov

Institute of Mathematics, Ukrainian Academy of Science, Tereshchenkivska Street 3, Kiev-4, Ukraine

Received 6 January 1994

Abstract. We develop a technique making it possible to handle the problem of separation of variables in nonlinear differential equations. Using it we obtain a number of new two-dimensional nonlinear wave equations admitting separation of variables and construct their exact solutions.

It is generally recognized that the method of separation of variables (SV) is one of the most universal and powerful means for study of linear partial differential equations (PDE). But the key idea of the method—the idea of reduction of PDE under study to several ordinary differential equations (ODE) can be applied to nonlinear equations as well. The classical example is the Hamilton–Jacobi equation [1, 6, 7]. Another example is provided by the nonlinear wave equation

$$\Box u = u_{x_0x_0} - u_{x_1x_1} = F(u) \tag{1}$$

which admits SV under $F(u) = \lambda \sin u$ [8] and under $F(u) = \lambda u \ln u$ [5].

In [3, 9] we suggested a new approach to the problem of separation of variables making it possible to study both linear and nonlinear equations. And what is more, the said method permits us to solve a classification problem, i.e. to describe PDE belonging to a given class which admits SV.

It is known that equation (1) separates into two second-order ODE iff $F(u) = (\lambda_1 + \lambda_2 u)\ln u, \lambda_i \in \mathbb{R}^1$ [5]. What is more, SV is possible in Cartesian coordinates x_0, x_1 only. Applying the technique developed in [3, 9] one can establish that the same result holds when separating PDE (1) into the first- and the second-order ODE.

In the present paper we study the case when nonlinear wave equation (1) admits separation into two first-order ODE. We obtain a number of new nonlinearities $F(u)$ permitting SV in (1), $F(u) = \lambda \sin u$, $F(u) = \lambda u \ln u$ being particular cases of the obtained formulae.

Consider the following first-order ODE:

$$\dot{\phi}_i(\omega_i) = R_i(\phi_i(\omega_i)) \qquad i = 1, 2 \tag{2}$$

where R_1, R_2 are some smooth functions, and the overdot denotes differentiation with respect to the corresponding argument.

0305-4470/94/090291+07$19.50

Definition. Ansatz

$$u(x) = f(x, \phi_1(\omega_1(x)), \phi_2(\omega_2(x))) \tag{3}$$

determines a solution of PDE (1) with separated variables in coordinates $\omega_1(x), \omega_2(x)$ if substitution of (3) into (1) with subsequent exclusion of the derivatives $\ddot{\phi}_i$, $\dot{\phi}_i$ according to formulae (2) yields an identity with respect to the variables ϕ_1, ϕ_2.

Let us note that standard approach to SV implies that the solution with separated variables is looked for either in the form $u(x) = a(x)\phi_1(\omega_1)\phi_2(\omega_2)$ (multiplicative SV) or in the form $u(x) = a(x) + \phi_1(\omega_1) + \phi_2(\omega_2)$ (additive SV) [6]. The principal idea of our approach is that the form of the ansatz for $u(x)$ should not be fixed *a priori*. The choice of functions f, ω_1, ω_2 must be determined by the form of the nonlinearity $F(u)$.

In the following, we study the case when in (2), (3) $\omega_1 = x_0, \omega_2 = x_1, f = g(\phi_1(x_0) + \phi_2(x_1))$ i.e. the solution of equation (1) is searched for in the form

$$u(x) = g(\phi_1(x_0) + \phi_2(x_1)) \qquad \dot{g} \neq 0. \tag{4}$$

Substituting (4) into (1) and excluding $\ddot{\phi}_i, \dot{\phi}_i, i = 1, 2$ according to formulae (2) we get

$$\dot{g}(R_1\dot{R}_1 - R_2\dot{R}_2) + \ddot{g}(R_1^2 - R_2^2) = F(g). \tag{5}$$

After redenoting $Q_i = \frac{1}{2}R_i^2$, $i = 1, 2$ we rewrite (5) in the following way:

$$\dot{g}(\dot{Q}_1 - \dot{Q}_2) + 2\ddot{g}(Q_1 - Q_2) = F(g(\phi_1 + \phi_2)). \tag{6}$$

Now, we have to split equality (6) with respect to independent variables ϕ_1, ϕ_2. For this we act on (6) by the operator $\partial/\partial\phi_1 - \partial/\partial\phi_2$, whence

$$\dot{g}(\ddot{Q}_1 + \ddot{Q}_2) + 2\ddot{g}(\dot{Q}_1 + \dot{Q}_2) = 0. \tag{7}$$

Consider the first case $\dot{Q}_1 + \dot{Q}_2 = 0$. Then the equality

$$\dot{Q}_1(\phi_1) = -\dot{Q}_2(\phi_2) = \lambda \qquad \lambda = \text{const} \tag{8}$$

holds.

Integration of ODE (8) gives

$$Q_1 = \lambda\phi_1 + C_1 \qquad Q_2 = -\lambda\phi_2 + C_2 \qquad C_1, C_2 \subset \mathbb{R}^1.$$

If one substitutes the result obtained into reduced ODE (2) and integrates these, then one obtains the following equalities:

$$\phi_1 + \phi_2 = \tilde{\lambda}((x_0 + \tilde{C}_1)^2 - (x_1 + \tilde{C}_2)^2) \tag{9}$$

or

$$\phi_1 + \phi_2 = \tilde{C}_1 x_0 + \tilde{C}_2 x_1 + C_3. \tag{10}$$

It means that under $\dot{Q}_1 + \dot{Q}_2 = 0$ the above described procedure yields group-invariant solutions of PDE (1) which have been studied in detail (see, e.g. [2]) and are not considered in the following.

Suppose now that inequality $\dot{Q}_1+\dot{Q}_2 \neq 0$ holds. Then (7) can be rewritten in equivalent form

$$\frac{-2\ddot{f}}{\dot{f}} = \frac{\ddot{Q}_1+\ddot{Q}_2}{\dot{Q}_1+\dot{Q}_2}.$$

Acting on the above relation by the operator $\partial/\partial\phi_1 - \partial/\partial\phi_2$, we get

$$(\dddot{Q}_1 - \dddot{Q}_2)(\dot{Q}_1+\dot{Q}_2) - \ddot{Q}_1^2 + \ddot{Q}_2^2 = 0. \tag{11}$$

Next, differentiating (11) with respect to ϕ_1 and ϕ_2, we arrive at the equality

$$Q_1^{(IV)}\ddot{Q}_2 - \ddot{Q}_1 Q_2^{(IV)} = 0. \tag{12}$$

The case 1. $\ddot{Q}_1\ddot{Q}_2 \neq 0$. Then from (12) it follows that

$$Q_1^{(IV)}/\ddot{Q}_1 = Q_2^{(IV)}/\ddot{Q}_2 = \lambda = \text{const}. \tag{13}$$

Integration of ODE *(13) yields*

$$\ddot{Q}_1 = \lambda Q_1 + \beta_1\phi_1 + \gamma_1$$

$$\ddot{Q}_2 = \lambda Q_2 + \beta_2\phi_2 + \gamma_2$$

where β_i, γ_i are arbitrary constants.

Further, one has to consider the subcases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ separately.

The subcase 1.1. $\lambda = -\alpha^2 < 0$. Then the general solution of ODE (13) is given by the formulae

$$Q_i = C_i\cos\alpha\phi_i + D_i\sin\alpha\phi_i + A_i\phi_i + B_i \tag{14}$$

where C_i, D_i, A_i, B_i, $i = 1, 2$ are arbitrary constants.

Substituting (14) into (11) we have

$$C_2^2 + D_2^2 - C_1^2 - D_1^2 + (A_1+A_2)(C_1\sin\alpha\phi_i - D_1\cos\alpha\phi_1 - C_2\sin\alpha\phi_2 + D_2\cos\alpha\phi_2) = 0. \tag{15}$$

If in (14) $C_i = 0$, $D_i = 0$, then integration of the corresponding ODE (2) gives formulae (9), (10). Otherwise, (15) is equivalent to the following equalities:

$$C_1^2 + D_1^2 = C_2^2 + D_2^2 \qquad A_1 = -A_2 = A$$

whence

$$Q_1 = C\cos(\alpha\phi_1 + \alpha_1) + A\phi_1 + B_1$$

$$Q_2 = C\cos(\alpha\phi_2 + \alpha_2) - A\phi_2 + B_2. \tag{16}$$

On making transformation $\phi_i \to \phi_i - \alpha_i\alpha^{-1}$ one can choose in (16) $\alpha_1 = \alpha_2 = 0$, i.e.

$$Q_1 = C\cos\alpha\phi_1 + A\phi_1 + B_1$$

$$Q_2 = C\cos\alpha\phi_2 - A\phi_2 + B_2. \tag{17}$$

Substituting (17) into (7) we get an equation for $g = g(\phi_1 + \phi_2)$

$$\ddot{g}(\dot{g})^{-1} = -\frac{\alpha}{2}(\cos\alpha\phi_1 + \cos\alpha\phi_2)(\sin\alpha\phi_1 + \sin\alpha\phi_2)^{-1} = -\frac{\alpha}{2}\operatorname{cotan}\left(\frac{\alpha}{2}(\phi_1 + \phi_2)\right).$$

General solution of the above equation reads

$$g = q_1 \ln\tan\frac{\alpha}{4}(\phi_1 + \phi_2) + q_2$$

where q_1, q_2 are arbitrary constants.

Without loosing generality we can choose $q_1 = 1$, $q_2 = 0$ i.e.

$$g = \ln\tan\frac{\alpha}{4}(\phi_1 + \phi_2).$$

After rescaling variables ϕ_1, ϕ_2 we get $\alpha = 4$. Substitution of $g = \ln\tan(\phi_1 + \phi_2)$ into (6) gives an explicit form of the function $F(g)$

$$F(g) = 4A(\cosh g - (\sinh 2g)\tan^{-1} e^g) + 4(B_1 - B_2)\sinh 2g.$$

Thus, we have established that the ansatz

$$u(x) = \ln\tan(\phi_1(x_0) + \phi_2(x_1))$$

reduces nonlinear wave equation

$$\Box u = 2A(\cosh u + (\sinh 2u)\tan^{-1} e^u) + 2(B_1 - B_2)\sinh 2u$$

to two first-order ODE

$$\dot{\phi}_1^2 = C\cos 4\phi_1 + A\phi_1 + B_1$$

$$\dot{\phi}_2^2 = C\cos 4\phi_2 - A\phi_2 + B_2.$$

The remaining cases are studied in a similar way. That is why we adduce only final results—ansatzes for $u(x)$ corresponding nonlinear PDE (1) and reduced equations.

The subcase 1.2. $\lambda = 0$.

$$u(x) = \ln(\phi_1(x_0) + \phi_2(x_1))$$

$$\Box u = Ae^u + 2(D_2 - D_1)e^{-2u}$$

$$\dot{\phi}_1^2 = 2A\phi_1^3 + B\phi_1^2 + C\phi_1 + D_1$$

$$\dot{\phi}_2^2 = -2A\phi_2^2 + B\phi_2^2 - C\phi_2 + D_2$$

where A, B, C, D_1, D_2 are arbitrary constants.

The subcase 1.3. $\lambda = \alpha^2 > 0$.

$$(1)\ u(x) = \ln\tanh(\phi_1(x_0) + \phi_2(x_1))$$

$$\Box u = 2A((\sinh 2u)\tan^{-1} e^u - \sinh u) + 2(B_1 - B_2)\sinh 2u$$

$$\dot\phi_1^2 = C\cosh 4\phi_1 + A\phi_1 + B_1$$

$$\dot\phi_2^2 = C\cosh 4\phi_2 - A\phi_2 + B_2$$

$$(2)\ u(x) = 2\tan^{-1}\exp(\phi_1(x_0) + \phi_2(x_1))$$

$$\Box u = A\left(2\sin u + (\sin 2u)\ln\tan\frac{u}{2}\right) + (B_1 - B_2)\sin 2u$$

$$\dot\phi_1^2 = C\sinh 2\phi_1 + 2A\phi_1 + 2B_1$$

$$\dot\phi_2^2 = C\sinh 2\phi_2 - 2A\phi_2 + 2B_2$$

$$(3)\ u(x) = \exp(\phi_1(x_0) + \phi_2(x_1))$$

$$\Box u = Au\ln u + (A + B_1 - B_2)u$$

$$\dot\phi_1^2 = C_1 e^{-2\phi_1} + A\phi_1 + B_1$$

$$\dot\phi_2^2 = C_2 e^{-2\phi_2} - A\phi_2 + B_2. \qquad (18)$$

Here, A, B_i, C, C_i are arbitrary constants.

The case 2. $\ddot Q_1 = 0$, $\ddot Q_2 \neq 0$. This case leads to formulae (18) with $C_1 = 0$.

The case 3. $\ddot Q_1 \neq 0$, $\ddot Q_2 = 0$. This case leads to formulae (18) with $C_2 = 0$.

The case 4. $\ddot Q_1 = \ddot Q_2 = 0$. This case yields for $\phi_1 + \phi_2$ expression of the form (9) or (10).

We say that equation (1) admits non-trivial SV if expression $\phi_1(x_0) + \phi_2(x_1)$ cannot be represented in the form (9) or (10). So we have proved the assertion.

Theorem. PDE (1) admits non-trivial SV in the class of functions (1) iff it is locally equivalent to one of the following nonlinear wave equations:

$$(1)\ \Box u = \lambda_1(\cosh u + (\sinh 2u)\tan^{-1} e^u) + \lambda_2 \sinh 2u \qquad (19)$$

$$(2)\ \Box u = \lambda_1 e^u + \lambda_2 e^{-2u} \qquad (20)$$

$$(3)\ \Box u = \lambda_1(\sinh u - (\sinh 2u)\,\mathrm{arctanh}\, e^u) + \lambda_2 \sinh 2u \qquad (21)$$

$$(4)\ \Box u = \lambda_1\left(2\sin u + (\sin 2u)\ln\tan\frac{u}{2}\right) + \lambda_2 \sin 2u \qquad (22)$$

$$(5)\ \Box u = \lambda_1 u + \lambda_2 u \ln u \qquad (23)$$

where λ_1, λ_2 are arbitrary constants.

Note. It is well known that the nonlinear wave equation (1) admits Lie–Bäcklund operators, whose coefficients do not depend on x_0, x_1, iff $F(u) = \lambda \sin u$ or $F(u) = \lambda_1 e^u + \lambda_2 e^{-2u}$ [4]. Evidently, equations (1) with such $F(u)$ are particular cases of PDE (20), (22).

In conclusion, we adduce exact solutions of nonlinear PDE (19)–(23) obtained by integration of the corresponding reduced ODE:

$$(1)\ u(x) = \ln\tan(\phi_1(x_0) + \phi_2(x_1))$$

$$\int^{\phi_1(x_0)} (C\cos 4\tau + A\tau + B_1)^{-1/2}\mathrm{d}\tau = x_0$$

$$\int^{\phi_2(x_1)} (C\cos 4\tau - A\tau + B_2)^{-1/2}\mathrm{d}\tau = x_1$$

where C, A, B_1, B_2 are arbitrary constants satisfying relations $A = \lambda_1/2$, $B_1 - B_2 = \lambda_2/2$;

$$(2)\ u(x) = \ln(\phi_1(x_0) + \phi_2(x_1))$$

$$\int^{\phi_1(x_0)} (2A\tau^3 + B\tau^2 + C\tau + D_1)^{-1/2}\mathrm{d}\tau = x_0$$

$$\int^{\phi_2(x_1)} (-2A\tau^3 + B\tau^2 - C\tau + D_2)^{-1/2}\mathrm{d}\tau = x_1$$

where A, B, C, D_1, D_2 are arbitrary constants satisfying relations $A = \lambda_1$, $D_2 - D_1 = \lambda_2/2$;

$$(3)\ u(x) = \ln\tanh(\phi_1(x_0) + \phi_2(x_1))$$

$$\int^{\phi_1(x_0)} (C\cosh 4\tau + A\tau + B_1)^{-1/2}\mathrm{d}\tau = x_0$$

$$\int^{\phi_2(x_1)} (C\cosh 4\tau - A\tau + B_2)^{-1/2}\mathrm{d}\tau = x_1$$

where C, A, B_1, B_2 are arbitrary constants satisfying relations $A = \lambda_1/2$, $B_1 - B_2 = \lambda_2/2$;

$$(4)\ u(x) = 2\tan^{-1}\exp(\phi_1(x_0) + \phi_2(x_1))$$

$$\int^{\phi_1(x_0)} (C\sinh 2\tau + 2A\tau + 2B_1)^{-1/2}\mathrm{d}\tau = x_0$$

$$\int^{\phi_2(x_1)} (C\sinh 2\tau - 2A\tau + 2B_2)^{-1/2}\mathrm{d}\tau = x_1 \tag{24}$$

where C, A, B_1, B_2 are arbitrary constants satisfying relations $A = \lambda_1$, $B_1 - B_2 = \lambda_2 - \lambda_1$;

$$(5)\ u(x) = \exp(\phi_1(x_0) + \phi_2(x_1))$$

$$\int^{\phi_1(x_0)} (C_1 e^{-2\tau} + A\tau + B_1)^{-1/2}\mathrm{d}\tau = x_0$$

$$\int^{\phi_2(x_1)} (C_2 e^{-2\tau} - A\tau + B_2)^{-1/2}\mathrm{d}\tau = x_1$$

where C_1, C_2, A, B_1, B_2 are arbitrary constants satisfying relations $A = \lambda_1$, $B_1 - B_2 = \lambda_2 - \lambda_1$.

If we choose in (24) $A = 0$, we get a solution of the sine-Gordon equation obtained in [8].

Thus, we have obtained a number of new nonlinear wave equations of the form (1) admitting SV and constructed their exact solutions.

It should be noted that the above solutions cannot be constructed by using Lie symmetry of equations (19)–(23). We guess that a possible way for obtaining such solutions is to utilize conditional symmetry of the nonlinear wave equation (1) [2] but this is a topic of a future paper.

References

[1] Bagrov V G, Gitman D M and Ternov I M 1982 *Exact Solutions of Relativistic Wave Equations* (Novosibirsk: Nauka)

[2] Fushchych W I, Shtelen W M and Serov N I 1992 *Symmetry Analysis and Exact Solutions of Nonlinear Equations of Mathematical Physics* (Dordrecht: Reidel)

[3] Fushchych W I, Zhdanov R Z and Revenko I V 1993 *Dokl. Akad. Nauk. Ukrainy* **1** 27

[4] Ibragimov N Kh 1983 *Transformation Groups in Mathematical Physics* (Moscow: Nauka)

[5] Koornwinder T H 1980 *Lecture Notes in Mathematics* **810** 240

[6] Miller W 1977 *Symmetry and Separation of Variables* (Reading, MA: Addison-Wesley)

[7] Morse P and Feshbach H 1953 *Methods of Theoretical Physics* part I (New York: McGraw-Hill)

[8] Osborne A and Stuart A E G 1978 *J. Math. Phys.* **19** 1573

[9] Zhdanov R Z, Revenko I V and Fushchych W I 1993 *J. Phys. A: Math. Gen.* **26** 5959